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Progress in distributive homology:

from q -polynomial of rooted trees to Yang-Baxter homology

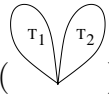
Abstract:

This is an extended abstract of the talk given at the Oberwolfach Workshop “Algebraic Structures in Low-Dimensional Topology”, 25 May – 31 May 2014. My goal was to describe progress in distributive homology from the previous Oberwolfach Workshop June 3 - June 9, 2012, in particular my work on Yang-Baxter homology; however I concentrated my talk on my recent discovery of q -polynomial of a rooted tree; the appropriate topic as my talk was on May 30, 2014, the 30 anniversary of the Jones polynomial, and the polynomial has its roots in the Kauffman bracket approach to the Jones polynomial.

We start with a long historical introduction beginning with Heinrich Kühn (1690-1769), Carl Leonhard Gottlieb Ehler (1685-1753), and Leonard Euler (1707-1783) and we argue that topology (*geometria situs*) started in Gdańsk (Danzig) about 1734. We mention the work of Celestyn Burstin (1888-1938) and Walter Mayer (1887-1948), (1929, distributivity) and Samuel Eilenberg (1913-1998) (homological algebra). We complete the historical summary by celebrating 30 years of the Jones polynomial (May 30, 1984, V.F.R.Jones wrote a letter to J.Birman announcing his construction of a new link polynomial). Thus it is appropriate to describe today a new simple invariant of rooted trees. Let T be a plane rooted tree then $Q(T) \in Z[q]$ is defined by the initial condition $T(\bullet) = 1$ and the recursion relation

$$Q(T) = \sum_{v \in L(T)} q^{r(T,v)} Q(T - v), \text{ where } L(T) \text{ is the set of leaves of } T,$$

and $r(T, v)$ is the number of edges of T to the right of the path connecting v with the root v_0 . For example $Q(\vee) = (1 + q) = [2]_q$ or more generally $Q(T_n) = [n]_q!$, where T_n is a star with n rays and $[n]_q = 1 + q + \dots + q^{n-1}$.

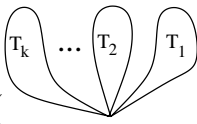
Theorem: Let $T_1 \vee T_2$ be the wedge (or root) product (). Then:

$$Q(T_1 \vee T_2) = \binom{E(T_1) + E(T_2)}{E(T_1)}_q Q(T_1) Q(T_2)$$

Proof: We proceed by induction on $E(T)$, with obvious initial case of $E(T_1) = 0$ or $E(T_2) = 0$. Let T be a rooted plane tree with $E(T_1)E(T_2) > 0$, then we have:

$$\begin{aligned}
Q(T) &= \sum_{v \in L(T)} q^{r(T,v)} Q(T-v) = \\
&\sum_{v \in L(T_1)} q^{r(T_1,v)+E(T_2)} Q((T_1-v) \vee T_2) + \sum_{v \in L(T_2)} q^{r(T_2,v)} Q(T_1 \vee (T_2-v)) \stackrel{\text{inductive assumption}}{=} \\
&\sum_{v \in L(T_1)} q^{r(T_1,v)+E(T_2)} \binom{E(T_1) + E(T_2) - 1}{E(T_1) - 1, E(T_2)}_q Q(T_1-v) Q(T_2) + \\
&\sum_{v \in L(T_2)} q^{r(T_2,v)} \binom{E(T_1) + E(T_2) - 1}{E(T_1), E(T_2) - 1}_q Q(T_1) Q(T_2-v) = \\
&Q(T_2) q^{E(T_2)} \binom{E(T_1) + E(T_2) - 1}{E(T_1) - 1, E(T_2)}_q \sum_{v \in L(T_1)} q^{r(T_1,v)} Q(T_1-v) + \\
&Q(T_1) \binom{E(T_1) + E(T_2) - 1}{E(T_1), E(T_2) - 1}_q \sum_{v \in L(T_2)} q^{r(T_2,v)} Q(T_2-v) = \\
&Q(T_1) Q(T_2) (q^{E(T_2)} \binom{E(T_1) + E(T_2) - 1}{E(T_1) - 1, E(T_2)}_q + \binom{E(T_1) + E(T_2) - 1}{E(T_1), E(T_2) - 1}_q) = \\
&Q(T_1) Q(T_2) \binom{E(T_1) + E(T_2)}{E(T_1), E(T_2)}_q \text{ as needed.}
\end{aligned}$$

Corollary:

- (i) If a plane rooted tree is a wedge of k trees () and

$$T = T_k \vee \dots \vee T_2 \vee T_1, \text{ then}$$

$$Q(T) = \binom{E_k + E_{k-1} + \dots + E_1}{E_k, E_{k-1}, \dots, E_1}_q Q(T_k) Q(T_{k-1}) \dots Q(T_1).$$

where $E_i = |E(T_i)|$ is the number of edges in T_i .

(ii) (State product formula)

$$Q(T) = \prod_{v \in V(T)} W(v),$$

where $W(v)$ is a weight of a vertex (we can call it a Boltzmann weight) defined by:

$$W(v) = \binom{E(T^v)}{E(T_{k_v}^v), \dots, E(T_1^v)}_q,$$

where T^v is a subtree of T with vertex v (part of T above v , in other words growing from v) and T^v can be decomposed into wedge of trees: $T^v = T_{k_v}^v \vee \dots \vee T_2^v \vee T_1^v$.

(iii) (change of a root). Let e be an edge of a tree T with endpoints v_1 and v_2 and E_1 be the number of edges on the v_1 part of the edge, and E_2 the number of edges of T on the v_2 side of e ;

Thus $T = \begin{array}{c} \tau_1 \quad \tau_2 \\ \diagdown \quad \diagup \\ v_1 \quad e \quad v_2 \end{array}$ and then $Q(T, v_1) = \frac{[E_1 + 1]_q}{[E_2 + 1]_q} Q(T, v_2)$.

Proof. (i) Formula z (i) follows by using several times the formula

$$Q(T_2 \vee T_1) = \binom{E(T_2) + E(T_1)}{E(T_2), E(T_1)}_q Q(T_2) Q(T_1),$$

as we have:

$$\begin{aligned} \binom{a_k + a_{k-1} + \dots + a_2 + a_1}{a_k, a_{k-1}, \dots, a_2, a_1}_q &= \binom{a_{k-1} + \dots + a_2 + a_1}{a_{k-1}, \dots, a_2, a_1}_q \binom{a_k + a_{k-1} + \dots + a_2 + a_1}{a_k, a_{k-1} + \dots + a_2 + a_1}_q = \dots \\ &= \binom{a_2 + a_1}{a_2, a_1}_q \binom{a_3 + a_2 + a_1}{a_3, a_2 + a_1}_q \binom{a_4 + a_3 + a_2 + a_1}{a_4, a_3 + a_2 + a_1}_q \dots \binom{a_k + a_{k-1} + \dots + a_2 + a_1}{a_k, a_{k-1} + \dots + a_2 + a_1}_q \end{aligned}$$

(ii) Formula (ii) follows by using (i) several times. \square

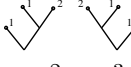
One can propose many modifications and generalizations of the polynomial $Q(T)$, for example, for a graph with a base point we can take the set (or the sum) over all spanning trees of $Q(T)$ but we propose below the one having close relation with knot theory.

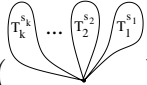
Let T be a plane rooted tree and $f : L(T) \rightarrow N$ a function from leaves of T to positive integers. We call f a delay function as our intuition is that a leaf with value k cannot be used before k th move. Formally $Q(T, f)$ is defined by recursive relation:

$$Q(T, f) = \sum_{v \in L_1(T)} q^{r(T, v)} Q(T - v, f_v),$$

where $L_1(T)$ is a set of leaves for which f is equal to 1. $f_v(u) = \max(1, f(u) - 1)$ if u is also a leaf of T , and it is equal to 1 if it is a new leaf of $T - v$.

Example. For a rooted tree with delay function the polynomial $Q(T)$ is not necessary a product of cyclotomic polynomials, the simplest example is

given by trees  with polynomials equal respectively $q(1 + q + 2q^2 + q^3)$ and $1 + 2q + q^2 + q^3$. There is however one special situation when we can give a simple closed formula: Consider the “delayed” tree $T = T_k^{s_k} \vee \dots \vee$

 $T_2^{s_2} \vee T_1^{s_1}$ (). That is we assume that whole blocks have constant delay function (the block T_i have leaves labelled s_i). We assume also, for convenience, that $s_1 = 1$, $s_1 \leq s_2 \leq E_1 + 1$, $s_2 \leq s_3 \leq E_2 + E_1 + 1, \dots$, $s_{k-1} \leq s_k \leq E_{k-1} + \dots + E_2 + E_1 + 1$ (here $E_i = |E(T_i)|$). Then

$$Q(T) = \binom{E_2 + E_1 - s_2 + 1}{E_2, E_1 - s_2 + 1}_q \binom{E_3 + E_2 + E_1 - s_3 + 1}{E_3, E_2 + E_1 - s_3 + 1}_q \dots \binom{E_k + \dots + E_1 - s_k + 1}{E_k, E_{k-1} + \dots + E_1 - s_k + 1}_q$$

$$Q(T_1)Q(T_2)\dots Q(T_k).$$

We didn't reach yet relations neither with knot theory nor with distributive structures; these should be left for the next occasion, however we finish the talk with one curious question and related observation. Consider a chain complex over a commutative ring k

$$\mathcal{C} : \dots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \rightarrow C_1 \xrightarrow{\partial_1} C_0$$

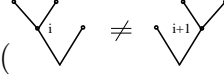
and assume that \mathcal{C} comes from a presimplicial module $\partial_n = \sum_{i=0}^n (-1)^i d_i$ where $0 \leq i \leq n$, $d_i d_j = d_{j-1} d_i$ for $i < j$. We ask whether it is useful (already used?) to consider q-version: $C_n^q = C_n \otimes_k Z[q]$ and the q-map $\partial_n^q = \sum_{i=0}^n q^i d_i$. Clearly (C_n^q, ∂_n^q) is not generically a chain complex but we can make another use of it. For example, we can identify x with $\partial^q(x)$, that

is to consider $(\bigoplus_{n \geq 0} C_n^q)/(x - \partial^q(x))$. Here an example which I learned from JP. Loday is very handy:

Consider presimplicial set (Y_n, d_i) where Y_n is the set of topological rooted

trees with n ordered leaves (topological means that $\nearrow = \searrow$). We define $d_i(T) = T - v_i$, where v_i is the i th leaf of T . We can also introduce degeneracy maps $s_i : Y_i \rightarrow Y_{i+1}$ planting \vee on the i th leaf. We check directly that:

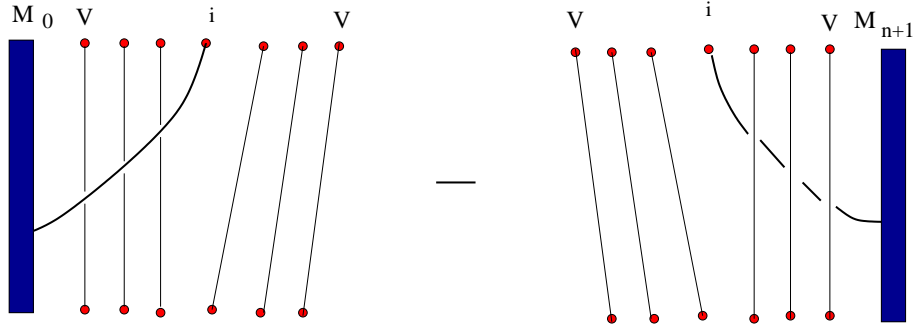
- (1) $d_i d_j = d_{j-1} d_i$ for $i < j$,
- (2') $s_i s_j = s_{j+1} s_i$ for $i < j$,
- (3) $d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ s_j d_{i-1} & \text{if } i > j + 1 \end{cases}$
- (4) $d_i s_i = d_{i+1} s_i = Id$



The condition $s_i s_i = s_{i+1} s_i$ does not hold ($\nearrow \neq \searrow$) so (Y_n, d_i, s_i) is not a simplicial set but only an almost simplicial set.

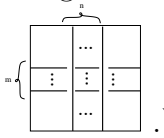
Now consider the quotient of the sum $(\bigoplus_{n \geq 0} Z[q]Y_n)/(x - \partial^q(x))$. It is a free $Z[q]$ module generated by \bullet (tree without edges). We compute inductively that for a tree with n leaves $T = [n]_q! \bullet$. It is not very sophisticated invariant so we can be glad that polynomial $Q(T)$ is more interesting.

Distributivity leads to another “incomplete” simplicial set, this time condition (4) does not hold, but this should be put aside for the next report which will discuss also a generalization of distributive homology: – Yang-Baxter homology.



Thank You

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